The Cholesky Decomposition - Part II

Gary Schurman MBE, CFA

June, 2012

A Cholesky matrix transforms a vector of uncorrelated (i.e. independent) normally-distributed random variates into a vector of correlated (i.e. dependent) normally-distributed random variates. In Part I we developed the mathematics for the Cholesky matrix by performing a true matrix LU decomposition. In Part II we will derive the Cholesky matrix via a simpler, more abbreviated approach. To this end we will use the correlation matrix from Part I which was...

$$\mathbf{C}=\begin{bmatrix}1.00 & 0.35 & 0.55\\ 0.35 & 1.00 & 0.25\\ 0.55 & 0.25 & 1.00\end{bmatrix} \tag{1}$$

In Part I we determined that the matrix decomposition took the form...

$$\mathbf{C}=\mathbf{L}\mathbf{L}^T \tag{2}$$

Note that matrix $\mathbf{L}$ is a lower triangular matrix and matrix $\mathbf{L}^T$ is its transpose. We can therefore define matrix $\mathbf{L}$ and matrix $\mathbf{L}^T$ as...

$$\mathbf{L}=\begin{bmatrix}a_{11} & 0 & 0\\ a_{21} & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \text{ ...and... } \mathbf{L}^T=\begin{bmatrix}a_{11} & a_{21} & a_{31}\\ 0 & a_{22} & a_{32}\\ 0 & 0 & a_{33}\end{bmatrix} \tag{3}$$

The dot product of matrix $\mathbf{L}$ and matrix $\mathbf{L}^T$ as defined in Equation (3) above is...

$$\mathbf{L}\mathbf{L}^T=\begin{bmatrix}a_{11}^2 & a_{11}a_{21} & a_{11}a_{31}\\ a_{11}a_{21} & a_{21}^2+a_{22}^2 & a_{21}a_{31}+a_{22}a_{32}\\ a_{11}a_{31} & a_{21}a_{31}+a_{22}a_{32} & a_{31}^2+a_{32}^2+a_{33}^2\end{bmatrix} \tag{4}$$

Using Equations (1) and (4) above we can now rewrite Equation (2) as...

$$\mathbf{C}=\mathbf{L}\mathbf{L}^T$$

$$\begin{bmatrix}1.00 & 0.35 & 0.55\\ 0.35 & 1.00 & 0.25\\ 0.55 & 0.25 & 1.00\end{bmatrix}=\begin{bmatrix}a_{11}^2 & a_{11}a_{21} & a_{11}a_{31}\\ a_{11}a_{21} & a_{21}^2+a_{22}^2 & a_{21}a_{31}+a_{22}a_{32}\\ a_{11}a_{31} & a_{21}a_{31}+a_{22}a_{32} & a_{31}^2+a_{32}^2+a_{33}^2\end{bmatrix} \tag{5}$$

As described in Part I our objective is to solve for matrix $\mathbf{L}$, which is the Cholesky matrix. We can accomplish this task via a bootstrapping method where we start at the top left corner of matrix $\mathbf{C}$ and work down each column of the lower triangular matrix solving for the elements of matrix $\mathbf{L}$ along the way. We therefore have the following equations that we will solve in order...

$$a_{11}^2=1.00 \tag{6}$$

$$a_{11}a_{21}=0.35 \tag{7}$$

$$a_{11}a_{31}=0.55 \tag{8}$$

$$a_{21}^2+a_{22}^2=1.00 \tag{9}$$

$$a_{21}a_{31}+a_{22}a_{32}=0.25 \tag{10}$$

$$a_{31}^2+a_{32}^2+a_{33}^2=1.00 \tag{11}$$

By solving for a_{11} in Equation (6) we get...

$$a_{11}^2=1.00$$

$$a_{11}=\sqrt{1.00}$$

$$a_{11}=1.00 \tag{12}$$

Using the result of Equation (12) above and solving for a_{21} in Equation (7) we get...

$$\begin{aligned} a_{11}a_{21} &= 0.35 \\ a_{21} &= 0.35 \div a_{11} \\ a_{21} &= 0.35 \end{aligned} \tag{13}$$

Using the result of Equation (12) above and solving for a_{31} in Equation (8) we get...

$$\begin{aligned} a_{11}a_{31} &= 0.55 \\ a_{31} &= 0.55 \div a_{11} \\ a_{31} &= 0.55 \end{aligned} \tag{14}$$

Using the result of Equation (13) above and solving for a_{22} in Equation (9) we get...

$$\begin{aligned} a_{21}^2 + a_{22}^2 &= 1.00 \\ a_{22}^2 &= 1.00 - a_{21}^2 \\ a_{22} &= \sqrt{0.8775} \\ a_{22} &= 0.9367 \end{aligned} \tag{15}$$

Using the result of Equations (13), (14) and (15) above and solving for a_{32} in Equation (10) we get...

$$\begin{aligned} a_{21}a_{31} + a_{22}a_{32} &= 0.25 \\ a_{32} &= (0.25 - a_{21}a_{31}) \div a_{22} \\ a_{32} &= (0.25 - 0.1925) \div 0.9367 \\ a_{32} &= 0.0614 \end{aligned} \tag{16}$$

Using the result of Equations (14) and (16) above and solving for a_{33} in Equation (11) we get...

$$\begin{aligned} a_{31}^2 + a_{32}^2 + a_{33}^2 &= 1.00 \\ a_{33}^2 &= 1.00 - a_{31}^2 + a_{32}^2 \\ a_{33} &= \sqrt{1.00 - 0.3025 - 0.0038} \\ a_{33} &= 0.8329 \end{aligned} \tag{17}$$

Using the results of the bootstrapping method as applied above, the Cholesky matrix, which is matrix **L** as defined in Equation (3) above, is therefore...

$$\text{Cholesky matrix} = \mathbf{L} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1.0000 & 0 & 0 \\ 0.3500 & 0.9367 & 0 \\ 0.5500 & 0.0614 & 0.8329 \end{bmatrix} \tag{18}$$